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A GENERALIZATION OF THE SOME-TIMES-POOL ESTIMATOR OF THE MEAN LIFE

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SUMMARY

Given two censored (type II) samples from one-parameter exponential models, a general class of estimators of the mean life of a component has been proposed. The some-times-pool estimator of the mean life (SPE) based on a preliminary test of significance is a member of this class. It is shown that estimators of this class including SPE are neither unbiased nor do they have uniformly minimum mean-squared error.

Keywords : Exponential model; Life test; Sometimes-pool procedure; Preliminary test.

Introduction

Ramkaran and Bhattacharya [5], had proposed a 'sometimes-pool' procedure for estimating the mean life in one parameter exponential model. Let $x_1, x_2, \ldots, x_r$ be an ordered sample from

$$f(x/\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x \geqslant 0,\ \theta > 0 \tag{1}$$

which is obtained by recording the first r failures out of n items placed on the test. Let $y_1, y_2, \ldots, y_k$ be another ordered sample from

$$f(y/\beta) = \frac{1}{\beta} e^{-y/\beta} \qquad y \geqslant 0,\ \beta > 0 \tag{2}$$

collecting sometime in the recent past by placing m items on a similar life test. If it is known for certain that $\theta \neq \beta$, then the best estimator of θ is given by T_r/r (Cf. [1]) where

$$T_r = \sum_{i=1}^{r} x_i + (n - r)\, x_r \tag{3}$$

Moreover, if it is known for certain that $\theta = \beta$, the pooled estimator $(T_r + S_k)(r + k)^{-1}$,
where

$$S_k = \sum_{j=1}^{k} y_j + (m - k)\, y_k \tag{4}$$

is better than T_r/r. When, however, it is not known whether or not the ratio of means $\lambda = \theta/\beta$ equals unity, it may still be possible to use any information provided by the statistic $\hat{\beta} = S_k/k$. In view of the uncertainity about the value of λ, we have proposed a sometimes-pool estimator of θ, after a preliminary test of significance of the hypothesis $H_0 : \lambda = 1$ for two alternatives viz $H_1 : \lambda < 1$ and $H_1 : \lambda > 1$. For the alternative $H_1 : \lambda < 1$, the proposed estimator can be described as follows :

$$\hat{\theta}_{SP} = \begin{cases} T_r/r & \text{if } Z < F_\alpha \\ (T_r + S_k)(r + k)^{-1}, & \text{otherwise} \end{cases} \tag{5}$$

where

$$Z = \frac{k\, T_r}{r\, S_k}. \tag{6}$$

Under the null hypothesis statistic Z has F distribution with $(2r, 2k)$ d.f. (Cf. [2]) and F_α represents its lower $100\alpha\%$ point. We have evaluated expressions for the bias and the mean-squared error and, with the help of numerical calculations, obtained that region in the parameter space in which the relative bias is within acceptable limits (i.e. the absolute value of relative bias is less than 5%) and the proposed estimator has smaller mean-squared-error than the usual estimator T_r/r. The purpose of present article is to define a general class of estimators of θ which contains the sometimes-pool estimator $\hat{\theta}_{SP}$ and to show the non-existence of an unbiased and uniformly minimum mean-squared-error estimator in this class.

2. The Generalized Estimator

Following Huntsberger [3], we propose the weighted estimator of θ as

$$\hat{\theta}_W(Z) = \phi(Z)\, \frac{T_r}{r} + \psi(Z) \cdot \frac{T_r + S_k}{r + k} \tag{7}$$

where $\phi(Z)$ $(= 1 - \psi(Z))$ is a function of the test statistic Z and provides weights for the unpooled and pooled estimators. The choice of weighting functions is restricted to the class of single valued functions of Z which are continuous except on a set of measure zero, which are defined for all Z, and which satisfy the following conditions :

$$0 \leqslant \phi(Z) = 1 - \psi(Z) \leqslant 1 \quad \text{for all } Z \geqslant 0 \tag{8}$$

It may be noted that the "sometimes-pool" estimator $\hat{\theta}_{SP}$ is merely a special case of $\hat{\theta}_W(Z)$ if $\phi(\cdot)$ is chosen to be a function which assumes the value zero with the acceptance of H_0 and unity otherwise.

In what follows, we shall prove that in the class of estimators given by (7), the only weighting function which leads to an unbiased estimator of θ is $\phi(Z) = 1$ for almost all $Z > 0$, that is, all estimators of type (7) other than the "never-pool" estimator are biased. This result is stated in the following theorem :

THEOREM 1 : *Provided that* $\lambda \neq 1$, the estimator $\hat{\theta}_W(Z)$ is unbiased for θ iff $\phi(\cdot) = 1$ almost everywhere.

Proof : The estimator at (7) can be written in the form

$$\hat{\theta}_W(Z) = \frac{T_r + S_k}{r + k} + \frac{k}{r + k}\,[\hat{\beta}(Z - 1)\,\phi(Z)] \tag{9}$$

so that

$$E[\hat{\theta}_W(Z)] = \frac{r\theta + k\beta}{r + k} + \frac{k}{r + k}\,E[\hat{\beta}(Z - 1)\,\phi(Z)] \tag{10}$$

In order to evaluate the second term on the right side of (10), we need the joint distribution of the statistics $Z = k\,T_r/r\,S_k$ and $\xi = \hat{\beta} = S_k/k$. From the knowledge of the distributions of T_r and S_k, we obtain the joint density

$$g^*(\xi, Z) = C\xi^{k+r-1}\,Z^{r-1}\,e^{-\left(\frac{k}{\beta} + \frac{rZ}{\theta}\right)\xi}\,;\; Z, \xi \geqslant 0 \tag{11}$$

where C is given by

$$C = \frac{r^r\,k^k}{(r-1)!\,(k-1)!\,\beta^k\,\theta^r} \tag{12}$$

Hence

$$E(\xi(Z - 1)\,\phi(Z)] = C\,(r + k)!\int_0^\infty \frac{(Z-1)\,Z^{r-1}\,\phi(Z)}{\left(\frac{k}{\beta} + \frac{rZ}{\theta}\right)^{k+r+1}}\,dZ \tag{13}$$

so that, we have for the bias

$$B[\hat{\theta}_W(Z)] = E[\hat{\theta}_W(Z)] - \theta$$

$$= \frac{k(\beta - \theta)}{r + k} + \left(\frac{k\lambda}{r}\right)^{k+1} \frac{\beta}{B(k, r)} \int_0^\infty \frac{(Z-1) Z^{r-1} \phi(Z)}{\left(Z + \frac{k\lambda}{r}\right)^{r+k+1}} dZ \tag{14}$$

If we replace $\phi(Z)$ by $1 - \psi(Z)$ in (14) and simplify, we obtain

$$B[\hat{\theta}_W(Z)] = -\int_0^\infty \frac{(Z-1)\psi(Z)}{\left(Z + \frac{k\lambda}{r}\right)^{r+k+1}} Z^{r-1} dZ \tag{15}$$

The simplification of (15) with the help of transformation $V = Z/\lambda$ leads to

$$B[\hat{\theta}_W(Z)] = \frac{-B(r, k)}{(k\lambda/r)^{k+1}} \int_0^\infty \eta(V) f_F(V)\, dV \tag{16}$$

where

$$\eta(V) = \frac{(\lambda V - 1)\psi(\lambda V)}{\left(1 + \frac{rV}{\lambda}\right)}$$

and $f_F(V)$ is the p.d.f. of F distribution with $(2r, 2k)$ degrees of freedom. Now, for unbiasedness of $\hat{\theta}_W(Z)$, the left hand side of (16) must vanish. Thus, the completeness of family of F distributions (see appendix) and unbiasedness of $\hat{\theta}_W(Z)$ together imply that $\eta(V) = 0$ for all $\lambda > 0$ and $V \geqslant 0$ except on a set of probability measure zero which in turn implies that $\psi(Z) = 0$ except on a set of probability measure zero.

3. Non-Existence of an Estimator with Uniformly Minimum Mean-Squared-Error

For the estimator $\hat{\theta}_W(Z)$ at (7) with the weighting function $\phi(Z)$, let us consider the mean-squared-error

$$\text{MSE}_\phi[\hat{\theta}_W(Z)] = E[\hat{\theta}_W(Z) - \theta]^2 \tag{17}$$

as a criterion of goodness of the estimator. Clearly, this MSE will be a function of the parameter $\gamma = (\theta, \beta)$. If there exists a weighting function $\phi(\cdot)$ such that

$$\text{MSF}_{\phi}[\hat{\theta}_W(Z)] \leqslant \text{MSE}_{\phi^*}[\hat{\theta}_W(Z)] \tag{18}$$

for every γ and for every other weighting function $\phi^*(\cdot)$, with strict inequality holding for atleast one γ and one $\phi^*(\cdot)$, then $\phi(Z)$ leads to a weighted estimator with uniformly minimum mean-squared-error. That such an estimator of type (7) does not exist, is shown in the following theorem.

THEOREM 2 : *For the class of estimators* (7), *there exists no weighting function* $\phi(\cdot)$ *such that*

$$\text{MSE}_{\phi}[\hat{\theta}_W(Z)] \leqslant \text{MSE}_{\phi^*}[\hat{\theta}_W(Z)] \tag{19}$$

for every γ *and every other weighting function* $\phi^*(\cdot)$.

Proof. The mean-squared-error

$$\text{MSE}_{\phi}[\hat{\theta}_W(Z)] = E[\hat{\theta}_W(Z) - \theta]^2$$
$$= E\left[\frac{T_r + S_k}{r+k} - \theta + \frac{k}{r+k}\hat{\beta}(Z-1)\phi(Z)\right]^2$$

can be evaluated by using the joint density $g^*(\xi, Z)$ at (11). After considerable simplification, we obtain

$$\text{MSE}_{\phi}[\hat{\theta}_W(Z)] = \frac{\theta^2}{r} + C^*\lambda^k\left\{k\int_0^\infty \frac{(Z-1)^2 Z^{r-1}\psi^2(Z)}{(k\lambda + rZ)^{r+k+2}}\,dZ\right.$$
$$+ 2\left(\frac{k+r}{k+r+1}\right)\int_0^\infty \frac{(Z-1)Z^{r-1}\psi(Z)\,dZ}{(k\lambda + rZ)^{r+k+1}}$$
$$- 2k\int_0^\infty \frac{(Z-1)^2 Z^{r-1}\psi(Z)\,dZ}{(k\lambda + rZ)^{k+r+2}}$$
$$- 2k\int_0^\infty \frac{(Z-1)Z^{r-1}\psi(Z)\,dZ}{(k\lambda + rZ)^{k+r+2}}$$
$$\left.- 2r\int_0^\infty \frac{(Z-1)Z^{r}\psi(Z)\,dZ}{(k\lambda + rZ)^{k+r+2}}\right\} \tag{20}$$

where one may recall that $\psi(Z) = 1 - \phi(Z)$, and C^* is as defined below :

$$C^* = \frac{k^{k+1}\, r^r\, (k + r + 1)!\, \theta^2}{(k + r)^2\, (k - 1)!\, (r - 1)!} \tag{21}$$

Recalling that the mean-squared-error of the "never-pool" estimator is $\text{MSE}(\theta) = \theta^2/r$, we consider, for variations in the mean ratio λ, the integral

$$I_\phi = \int_0^\infty \{\text{MSE}(\hat{\theta}) - \text{MSE}_\phi[\hat{\theta}_W(Z)]\}\, d\lambda$$

$$= C^* \frac{B(k + 1, r + 1)}{k^{k+1}\, r^{r+1}} \Big\{ - k \int_0^\infty \left(\frac{Z - 1}{Z}\right)^2 \psi^2(Z) dZ - 2(k + r) \cdot \int_0^\infty \left(\frac{Z - 1}{Z}\right) \psi(Z)\, dZ + 2k \int_0^\infty \left(\frac{Z - 1}{Z}\right)^2 \psi^2(Z)\, dZ + 2k \cdot \int_0^\infty \frac{(Z - 1)\, \psi(Z)}{Z^2}\, dZ + 2r \int_0^\infty \left(\frac{Z - 1}{Z}\right) \psi(Z)\, dZ \Big\} \tag{22}$$

which has been obtained after interchanging the order of integration with respect to Z and λ in the double integrals involved in I_ϕ and integrating out λ. It is now easy to verify that (22) reduces to

$$I_\phi = -\left(\frac{k}{k + 1}\right)^2 \theta^2 \int_0^\infty \left(\frac{Z - 1}{Z}\right)^2 \{1 - \phi(Z)\}^2\, dZ \tag{23}$$

assuring that the maximum value of the integral I_ϕ is zero. This maximum is attained if $\phi(\cdot) = 1$ almost everywhere, so that for any other weighting function we have $I_\phi < 0$. It is now clear that (19) cannot hold with $\phi^* = 1$ and ϕ-any other weighting function, since if it did then we would have $I_\phi \geqslant 0$ which is impossible by virtue of the immediately preceding statement. It only remains to see that $\phi^* = 1$ does not lead to an estimator with uniformly minimum mean-squared-error. That this is indeed so can be established by considering the case $\lambda = 1$ and the estimator $(T_r + S_k)/(r + k)$ which has the variance $\theta^2/(r + k) < \theta^2/r$, the variance for the case $\phi^* = 1$, and this completes the proof of the theorem.

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APPENDIX

THEOREM : *The family of F distributions is complete.*

Proof : Let $(X_{11}, X_{12}, \ldots, X_{1j}, \ldots, X_{1n_1})$ and $(X_{21}, X_{22}, \ldots, X_{2j}, \ldots, X_{2n_2})$ be independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Then the statistic

$S_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2$, $i = 1, 2$, is an unbiased estimator of σ_i^2. By M. N. Ghosh's general theorem on completeness for exponential families (Cf. Lehmann [4]), it follows that statistic (S_1^2, S_2^2) is complete. Thus,

$$\int_0^\infty \int_0^\infty G\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}\right) h\left(\frac{S_1^2}{\sigma_1^2}, \frac{S_2^2}{\sigma_2^2}\right) dS_1^2 \, dS_2^2 = 0 \tag{A.1}$$

implies that $G\left(\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}\right) = 0$ for all (S_1^2, S_2^2) and $\sigma_1^2, \sigma_2^2 > 0$ except on a set of probability measure zero. In (A.1), $h\left(\frac{S_1^2}{\sigma_1^2}, \frac{S_2^2}{\sigma_2^2}\right)$ is the joint probability density function of S_1^2/σ_1^2 and S_2^2/σ_2^2. Now, using the transformation $U = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$ and $W = S_1^2/\sigma_1^2$ in (A.1), we conclude that

$$\int_0^\infty \int_0^\infty G(U)\, h'(U, W)\, dU\, dW = 0 \tag{A.2}$$

implies that $G(U) = 0$ except on a set of probability measure zero. In (A.2), $h'(U, W)$ is the joint p.d.f. of U and W. By integrating out W in (A.2), we reach the conclusion that

$$\int_0^\infty G(U)\, h''(U)\, dU = 0 \tag{A.3}$$

implies that $G(U) = 0$ except on a set of probability measure zero. But $h''(U)$ is the p.d.f. of F distribution. Thus, the completeness of family of F distributions follows immediately from (A.3).